Trapezoids

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**Trapezoids**

 In every high school textbook of geometry, the trapezoid in included as one of the quadrilaterals to study (or investigate). In almost every one, it is defined as a quadrilateral with exactly one pair of parallel sides. In this talk, I will argue that the definition should be changed and that there is much more to the trapezoid than is given in the books. Almost every theorem about a trapezoid can be broken into two categories – those that are really about parallel sides and those that in some way incorporate a feature of a quadrilateral. In the history of geometry, many definitions have been changed, some from being exclusive to being inclusive. An exclusive definition is one that separates other objects from being in the same class. For example, in Euclid, an isosceles triangle is defined to be a triangle with exactly two equal sides. Thus, an equilateral triangle is not isosceles. Likewise, all his definitions of quadrilaterals are exclusive. During the next 2300 years, most of these have been changed. Now in every high school textbook, equilateral triangles are isosceles, rectangles and rhombi are parallelograms, and squares are rectangles and rhombi. There are two advantages to having inclusive definitions – (1) theorems for the more restricted case become corollaries for the more general case and (2) converses do not need to contain an “or” conclusion. By one construction, we can easily see that the parallelogram is a special case of a trapezoid. Take two points on each of two parallel lines. The quadrilateral formed by using these points is a trapezoid (or a parallelogram). Consequently, the definition can be strengthened by including it as such.

 So why is the definition maintained in the textbooks? I think primarily that many authors do not wish to go against standard terminology, most of the authors have not thought about the inconsistency in terminology, they are not actively engaged in discovering and proving theorems in geometry, and there are no converses for the trapezoid covered in high school geometry. Thus, for good mathematical reasons let’s change the definition to: **A trapezoid is a quadrilateral with at least one pair of parallel sides.** In the theorems that follow, some require that a pair of sides be non-parallel, but the parallel case follows as well, usually with little or no additional proof. While we are at this change, the same argument applies to the isosceles trapezoid and the rectangle. A simple construction shows that the rectangle is a special case of the isosceles trapezoid. How then can we define the isosceles trapezoid so that the rectangle is a special case. I offer a variety of different definitions. (1) An isosceles trapezoid is a cyclic trapezoid. (2) An isosceles trapezoid is a trapezoid with a pair of supplementary opposite angles. (3) An isosceles trapezoid is a trapezoid with the other pair of sides anti-parallel with respect to the parallel sides. (4) An isosceles trapezoid is a trapezoid with a pair of congruent base angles. (5) An isosceles trapezoid is a trapezoid with congruent diagonals.

 Trapezoids are connected to triangles naturally since a trapezoid can be formed by constructing a parallel through one side of the triangle to a second side of the triangle; thus, producing a triangle similar to the original triangle and a trapezoid.

 Here are a few other definitions that I offer.

The **diacenter** of a quadrilateral is the intersection point of the diagonals.

A **quord** of a quadrilateral is a segment with endpoints on two sides of a quadrilateral.

A **median** is a quord with endpoints the midpoints on opposite sides.

**Theorems in Elementary Geometry about Trapezoids [T]:**

The symbol [T] will be used to mean a trapezoid. For a trapezoid that is not parallelogram, the non-parallel sides will called **legs** and the parallel sides will be called **bases**.



1) The median joining the legs of a [T] is parallel to the bases.

2) The length of the median joining the legs of a [T] is the arithmetic average of the lengths of the bases.

3a) The area of a [T] is one half the height multiplied by the sum of the legs.

Combine this with Theorem 2 and a second formula is obtained.

3b) The area of a [T] is one half the height multiplied by the length of the median joining the legs.



4) The diacenter divides the diagonals of a [T] proportionally to each other and to the bases.

 $\frac{AO}{CO}=\frac{BO}{DO}=\frac{AB}{CD}$

5) Converse of (4): If the diacenter divides the diagonals proportionally, then the quadrilateral is a [T].

6) The triangles formed by the diacenter and the vertices of the legs of a [T] are equivalent (equal in area).

 area(ΔAOD) = area(ΔBOC)

7) Let the median joining the legs of a [T] intersect the diagonals at points P and Q.

 $PQ=\frac{\left|AB-CD\right|}{2}$

**A few theorems not seen in high school geometry books.**

The Other Median

8) The median determined by the midpoints of the bases of a [T] contains the diacenter.

9) (Converse of 8) If a median of a quadrilateral contains the diacenter, then the quadrilateral is a [T].



10) If W is on the median $\overbar{ST}$ of [T] ABCD with bases $\overbar{AB}$ and $\overbar{CD}$, then ΔAWD and ΔBWC are equivalent.

10a) If the area of ΔAWD and ΔBWC are equivalent, then W is on the median $\overbar{ST}$.

11) The median between the bases of a [T] bisects any quord parallel to the other median (or bases).

12) The locus of the midpoints of a quord parallel to the bases of a [T] is the median between the bases.

13) If $\overbar{NM}$ is the median between the bases $\overbar{AB}$ and $\overbar{CD}$ of a [T], then

 $NM^{2}= \frac{AD^{2}+BC^{2}}{2}- \frac{\left(AB-CD\right)^{2}}{4}$ and $NM^{2}= \frac{AC^{2}+BD^{2}}{2}- \frac{\left(AB+CD\right)^{2}}{4}$

14) The area of a [T] is equal to the product of the length of a non-parallel side times the altitude to it from the midpoint of the other non-parallel side.

15) In [T] the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the legs and twice the product of the lengths of the bases.

16) The sum of the squares of the lengths of the diagonals of a trapezoid is equal to the sum of the squares of the lengths of the legs and twice the product of the lengths of the bases.

17) Th [T] of maximum area inscribed in a segment of a circle is isosceles with 3 congruent sides. [Note: this cannot be done with compass and straight edge since it is equivalent to trisecting an angle.]

[from FG-M, Exercises in Geometrie, 6th Edition, #365]

16a) The diacenter bisects the quord between the legs of a [T]

through the diacenter.

16b) A quord between the legs of a [T] that is bisected by the diacenter is parallel to the bases.

16c) The length of the quord through the diacenter parallel to the bases of a [T] is equal to twice the product of the bases divided by the sum of the lengths of the bases.

$$UV=\frac{2AB∙CD}{AB+CD}$$

16d) $\frac{BV}{BC}=\frac{BO}{BD}=\frac{AO}{AC}=\frac{AU}{AD}=\frac{AB}{AB+CD}$ , $\frac{CV}{BC}=\frac{CO}{CA}=\frac{DU}{DA}=\frac{DO}{DB}=\frac{CD}{AB+CD}$



17) If $\overbar{EF}$ is a quord parallel to the bases $\overbar{AB} $and $\overbar{CD}$ of [T] ABCD, then

 $EF=\frac{AE∙DC+AB∙ED}{AD}= \frac{BF∙DC+AB∙FC}{BC}$